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A ZERO-ONE LAW FOR A CLASS OF MEASURES ON GROUPS

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ABSTRACT

Let $(E,\mathcal{B})$ be a measurable (non-abelian) group, α a measurable homomorphism on E and μ a law on $(E,\mathcal{B})$ satisfying $\mu = \mu_\alpha \nu_\alpha$ (with $\mu_\alpha = \mu \circ \alpha^{-1}$), for some law ν_α on $(E,\mathcal{B})$. Under some additional conditions ~~on μ, ν_α and α,~~ it is shown that, for every proper normal subgroup G of E and a in E, if $Ga \in \mathcal{B}^*_{(\mu)}$, the μ-completion of $\mathcal{B}$, then $\mu(Ga) = 0$ or 1. As a corollary, it is shown that this result yields all previously known 0-1 laws for stable, semistable, and quasistable laws on linear spaces as well as new 0-1 laws for other classes of infinitely divisible laws on linear spaces.

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The main purpose of this paper is to prove the

Theorem: We suppose that $(E,\mathcal{B})$ is a measurable (non-abelian) group (i.e. E is a non-abelian group, $\mathcal{B}$ is a σ-algebra of subsets of E such that the map $x \times x' \to xx'^{-1}$ is measurable relative to the σ-algebras $\mathcal{B} \times \mathcal{B}$ and $\mathcal{B}$) or that $(E,\mathcal{B})$ is a Hausdorff topological group with $\mathcal{B}$ its Borel σ-algebra. In the latter case all laws considered are assumed to be τ-regular. Let α be a measurable homomorphism from E into E (i.e. a measurable(relative to $\mathcal{B}$ and $\mathcal{B}$) map, preserving the group operation) and μ a law on $\mathcal{B}$; and let μ_α and $\mathcal{B}_\mu$ denote, respectively, the image of μ under α and the μ-completion of $\mathcal{B}$. We assume that μ_α is a left factor of μ (on $\mathcal{B}$ and hence on $\mathcal{B}_\mu$) and an n-th root of μ_α exists which divides the cofactor ν_α :

(1) $\mu = \mu_\alpha \nu_\alpha$, (1') $\nu_\alpha = \mu^{1/n} \nu'_\alpha$ or $\nu_\alpha = \nu'_\alpha \mu^{1/n}$.

Let G be a normal subgroup of E and assume that

(2) $\alpha(G) \subseteq G$, $\alpha(G^c) \subseteq G^c$, (2')* $\alpha^n I^{-1}(G^c) \subseteq G^c$,

for all $n = 1,2,\ldots,$ where α^n denotes the n-th iteration of α and G^c denotes the complement of G . Then, for any $a \in E$, we have

(3) $Ga \in \mathcal{B}_\mu \Rightarrow \mu(Ga) = 0$ or 1 .

(*) $(\alpha^n I^{-1})(x) = \alpha^n(x)\, x^{-1}$, for $x \in E$.

If $G = \{e\}$† and $\{a\} \in \mathcal{B}_\mu$, for all $a \in E$, or if G is closed, then we can, in the topological case, replace (1') and (2') by

$$(1'') \qquad \lim_n \alpha^n(x) = e \quad , \text{ for all } x \in E .$$

<u>Proof</u>. Assume $\mu(Ga) > 0$, we will show $\mu(Ga) = 1$. Let G_0 denote the group of right cosets Gx . Condition (2) assures us that α induces in G_0 an injective homomorphism. We denote the coset Gc by $\dot{c}$; if $\dot{c} \in \mathcal{B}_\mu$, then, from Fubini's theorem and (1) , we have (with $\nu \equiv \nu_\alpha$)

$$(4) \qquad \mu(\dot{c}) = \int_E \mu\{\alpha^{-1}(\dot{c}x^{-1})\}\, \nu(dx) \ .$$

In (4), $\alpha^{-1}(\dot{c}x^{-1}) \in \mathcal{B}_\mu$ and $g(x) = \mu\{\alpha^{-1}(\dot{c}\, x^{-1})\}$ is $\mathcal{B}$-measurable off a set $N \in \mathcal{B}$ with $\nu(N) = 0$.

Let $A = \{\dot{c}_k \equiv Gc_k \colon k = 1,\ldots,K\}$ form the finite set of all distinct μ-measurable right cosets of G such that $\mu(\dot{c}_k)$ is maximum (among all μ-measurable right cosets). We will show $K = 1$ by showing that $K \geq 2$ leads to a contradiction. From (4) and the maximality of $\mu(\dot{c}_k)$, one can choose distinct cosets $\dot{b}_i^{-1}$, $i = 1,\ldots,I$, $I \leq K$, such that $\nu(\cup \dot{b}_i^{-1}) = 1$ and $\alpha^{-1}(\dot{c}_k b_i) = \dot{c}_{ki} \in A$, with $\dot{c}_{ki}$ distinct (for fixed i) (we don't know that $\dot{b}_i^{-1}$ is ν-measurable). Denote by Δ the set of symmetric (distinct)

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† This case, when μ is infinitely divisible and E is a topological space, comes up in the study of atoms of these laws [1]; the results of [1] trivially extend from the cylindrical σ-algebra to $\mathcal{B}$ when μ is τ-regular.

elements $\dot{c}_k\dot{c}_{k'}^{-1}$, $k \neq k'$, say $\Delta \equiv \{\delta_\ell : \ell = 1,\ldots,L\}$ with $1 \leq L$ (as $K \geq 2$) . Then since $\dot{c}_k \neq \dot{c}_{k'}$ implies $\dot{c}_{ki} \neq \dot{c}_{k'i}$ and $\alpha^{-1}(\dot{c}_k\dot{c}_{k'}^{-1}) = \alpha^{-1}(\dot{c}_k b_i b_i^{-1}\dot{c}_{k'}^{-1}) = \dot{c}_{ki}\dot{c}_{k'i}^{-1}$, it follows that

$$\alpha^{-1}(\delta_\ell) = \delta_{\ell'} \in \Delta .$$

Thus α^{-1} is a permutation in Δ ; hence α^n has a fixed point, say $\dot{b}$, for some $n \leq L$. In E this is written : $\alpha^n(b) = gb$, $g \in G$; thus $\alpha^n I^{-1}(b) \in G$. Then (2') implies $b \in G$; thus $\dot{b} = \varepsilon$, the unit element of G_0 , but $\varepsilon \notin \Delta$. This is a contradiction; hence $K = I = 1$. Hence $\nu(\dot{c}_1) = 1$; but then (1') and $\mu(\dot{a}) > 0$ imply $\mu(\dot{a}) = 1$. In the case that $G = \{e\}$ and in order to replace (1') by (1") , we make use of the following:

<u>Lemma</u>. Let G_0 be a Hausdorff topological group with unit element ϵ μ a law (of a random variable X) on the Borel σ-algebra $\mathcal{B}_0$ of G_0 . We assume μ is τ-regular and has nonempty support (if G_0 is metrizable) and α is as given above from E into E . Then the conditions

$$(5) \quad \alpha(X) \overset{\text{law}}{=} Xb \qquad \text{and} \qquad (1") \quad \alpha^n(x) \underset{n\to\infty}{\to} \epsilon \ , \quad \text{for all } x ,$$

imply that μ is improper : $X = a$ a.s. with $\alpha(a) = ab$.

<u>Proof</u>. Let V be a symmetric neighborhood of ϵ and $\mu(Va) = \eta > 0$. Then (1") implies that

$$\{x : \alpha^n(x) \in V , \ \text{all} \ n \geq m\} \uparrow G_0 \ \text{ when } \ m \uparrow \infty \ .$$

On the otherhand (5) gives

$$\alpha^n(X) \overset{\text{law}}{=} X\, b_n \quad , \text{ with } b_n = \alpha^{n-1}(b)\ldots\alpha(b)b \;.$$

Thus, we have $P\{\alpha^n(X)b_n^{-1} = X \in Vb_n^{-1}\} = P\{\alpha^n(X) \in V\} \to \bullet$.

Hence, Vb_n^{-1} intersects Va for all $n \geq n_0$ sufficiently large; thus $b_n^{-1} \in V^2a$ and $P\{X \in V^3a\} \geq P\{X \in Vb_n^{-1}\} \to 1$. Hence the τ-regularity of μ or the weaker hypothesis that its support is nonempty (if G_0 is metrisable) assures $X = a$ a.s. That $\alpha(a) = ab$ is clearly true.

Remark. Even if E is a Hausdorff topological group, G nonclosed (which is the interesting case) does not assure that G_0 is such a group. Then it is very possible that, with the atom $\dot{a}$ of G_0 removed, there remains a single atom, for the law μ is zero on the points of G_0 . In this case the relation $\alpha(X) \overset{\text{law}}{=} Xb$ is possible*, and yet $\mu(\dot{a}) = 1$ is not necessarily true (: the G-cylinders of E , belonging to $\mathcal{B}_\mu$, except Ga , reduce to the countable union of classes which have μ-measure 0 and to the complements of these unions which have μ-measure 1 , and are equivalent to the above mentioned atom). Thus hypothesis (1'), which appears artificial except in the case when μ is infinitely divisible, seems indispensible.

Corollary. Let E be a measurable ($x \times x' \to x + x'$ and $t \times x \to tx, t$ real, are measurable) or a Hausdorff topological vector space. Let $0 < \alpha < 1$,

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* $G_0 \setminus \{\dot{a}\}$ can also, for example, consist of two equal atoms which can be exchanged under the action of α or two unequal (and thus) invariant atoms. By hypothesis these atoms are not reduced to the points of G_0 with zero μ measure.

$\alpha(\cdot)$ the map $\alpha(x) = \alpha \cdot x$, and $\ell(\alpha)$ the field generated in R by α . Let G be a additive subgroup of E . We retain for μ the conditions (1) and (1') (and only (1) if E is a Hausdorff topological vector space and $G = \{0\}$). Then $G'+a \in \mathcal{B}_\mu$ (with $G' = \ell(\alpha)G$) implies

$$(6) \qquad \mu(G' + a) = 0 \text{ or } 1 .$$

Thus, $G \in \mathcal{B}$ also implies (6) .

Proof. G' satisfies (2) and (2') .

This applies to the quasi-stable, semi-stable [5], and stable laws [4] without passing to symmetrisation or to centering of laws, but we can not deduce $G \in \mathcal{B}_\mu \Rightarrow \mu(G+a) = 0$ or 1 , a property known for a centered Gaussian law (: stable of exponent 2). In fact for α rational the condition (2') along with $\alpha G = G$ (note that for any bijective map α on E , not necessarily a homomorphism, the condition $\alpha(G) = G$ is equivalent to (2)) imply that G is a vector space over the field Q of rationals and thus the theorem applies only to certain groups.

Other examples: Let E be a measurable or a Hausdorff topological vector space, $\mu = P(M)$ be a Poisson type infinitely divisible law on E , and α a 1-1 bounded linear operator on E . The condition

$$(7) \qquad \alpha(M) \equiv M_\alpha \leq \frac{n-1}{n} M$$

(for large enough n) implies that $\mu_\alpha = P(M_\alpha)$ satisfies (1) and (1') . Then every $G + a$ satisfying (2) and (2') is of μ-measure 0 or 1 (but the condition (2') is difficult to interpret and verify). Thus, when

Then every $G + a$ satisfying (2) and (2') is of μ-measure 0 or 1 (but the condition (2') is difficult to interpret and verify). Thus, when $\alpha\{\cdot\}$ is multiplication by a number $\alpha \in (0,1)$, one obtains 0-1 laws for a certain class of self-decomposible laws (see [2]) as well as for many others (e.g. Radon α-decomposible laws $P(M)$ with M satisfying (7)). An analogus remark applies (with α 1-1 linear and α and α^{-1} bounded) to the laws of class L studied by Urbanik in [3].

REFERENCES

[1] Albert Tortrat, Atomes et lois indéfiniment divisibles dan un espace vectoriel, Ann. Inst. H. Poincarè XIV, 3, 1978, 345-349.

[2] Dong M. Chung, Characterization of self decomposible probability measures on LCTVS, J. Korean Math. Soc., 15-2 (1979), 139-148.

[3] K. Urbanik, Lévy's probability measures on Banach spaces, Studia Mathematica, LXIII (1978), 283-308.

[4] R.M. Dudley and M. Kantor, Zero-one laws for stable measures, Proc. Amer. Math. Soc., 45 (1974), 245-252.

[5] D. Louie, Balram S. Rajput and Albert Tortrat, A zero-one dichotomy theorem for r-semistable laws on infinite dimensional linear spaces, Sankhyā, (1979), (in press).

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Abstract(continued)

it is shown that this result yields all previously known 0-1 laws for stable, semistable, and quasistable laws on linear spaces as well as new 0-1 laws for other classes of infinitely divisible laws on linear spaces.